PMM U.S.S.R.,Vol.52,No.6,pp.715-720,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

ADMISSIBILITY OF SIMPLIFIED EQUATIONS IN THE DYNAMICS OF GYROSCOPIC SYSTEMS*

L.K. KUZ'MINA

Based on the asymptotic approach of /1/, rigorous mathematical methods are used to single out some known simplified models from the theory of gyroscopic systems and to prove that they may legitimately be employed to solve problems in dynamics (including stability problems). The initial system is of the singularly perturbed type /2/. The use of methods from stability theory /3, 4/ yields conditions under which transition to a simplified (computational) model is permissible. Several papers have been devoted to the solution of such problems for singularly perturbed equations /5/ by methods of Lyapunov theory.

1. Consider a gyroscopic system whose state is determined by n generalized (defining) Lagrange coordinates. The differential equations of perturbed motion of this system, written in Lagrangian form, are

$$d a \dot{\mathbf{q}}_M/dt + (b^\circ + g^\circ)\dot{\mathbf{q}}_M = \mathbf{Q}_M'(\mathbf{q}_M) + \mathbf{Q}_M''(\mathbf{q}_M, \dot{\mathbf{q}}_M) \tag{1.1}$$
$$d\mathbf{q}_M/dt = \dot{\mathbf{q}}_M$$

Here $\mathbf{q}_M$ is the n-dimensional vector of mechanical generalized coordinates, $a(\mathbf{q}_M)$ is the symmetric matrix of the positive-definite quadratic form representing the kinetic energy of the system, $b(\mathbf{q}_M)$ is the symmetric matrix of the positive-semidefinite quadratic form occurring in the decomposition of the dissipative function of viscous friction forces, $g(\mathbf{q}_M)$ is the skew-symmetric matrix of gyroscopic coefficients, $\mathbf{Q}_M'(\mathbf{q}_M) = -e^\circ\mathbf{q}_M$, where $e = e(\mathbf{q}_M)$ is the square matrix of (potential and non-potential) forces, depending on the generalized coordinates, and $\mathbf{Q}_M''$ collects all the non-linear terms, $\mathbf{Q}_M''(0, 0) = 0$.

We shall assume that all functions in (1.1) are jointly holomorphic in all the variables (in a certain region).

The simplified model for a system of type (1.1), known in the literature as the "precessional" model, appears in various versions:

$$(b^\circ + g^\circ)\dot{\mathbf{q}}_M = \mathbf{Q}_M'(\mathbf{q}_M) + \overline{\mathbf{Q}}_M''(\mathbf{q}_M, \dot{\mathbf{q}}_M), \quad d\mathbf{q}_M/dt = \dot{\mathbf{q}}_M \tag{1.2}$$

$$g^\circ\dot{\mathbf{q}}_M = \mathbf{Q}_M'(\mathbf{q}_M) + \overline{\mathbf{Q}}_M''(\mathbf{q}_M, \dot{\mathbf{q}}_M), \quad d\mathbf{q}_M/dt = \dot{\mathbf{q}}_M \tag{1.3}$$

$$(b_1^\circ + g_1^\circ)\dot{\mathbf{q}}_M = \mathbf{Q}_1'(\mathbf{q}_M) + \overline{\mathbf{Q}}_1''(\mathbf{q}_M, \dot{\mathbf{q}}_M) \tag{1.4}$$
$$d\tilde{a}_2\dot{\mathbf{q}}_M/dt + (b_2^\circ + g_2^\circ)\dot{\mathbf{q}}_M = \mathbf{Q}_2'(\mathbf{q}_M) + \overline{\mathbf{Q}}_2''(\mathbf{q}_M, \dot{\mathbf{q}}_M)$$
$$d\mathbf{q}_M/dt = \dot{\mathbf{q}}_M$$

(see /6, 7/, /6, 8/ and /9, 10/, respectively. Here

$$\mathbf{q}_M = \| \mathbf{q}_1, \mathbf{q}_2 \|^T, \quad a = \| a_1, a_2 \|^T, \quad b = \| b_1, b_2 \|^T,$$
$$g = \| g_1, g_2 \|^T, \quad \mathbf{Q}_M = \| \mathbf{Q}_1, \mathbf{Q}_2 \|^T$$

where $\mathbf{q}_1$ is the n_1-dimensional vector of generalized mechanical coordinates defining the position of the gyroscope suspensions (relative to the object, the platform stabilized), a_i, b_i, g_i $(i = 1, 2)$ are minors of the appropriate orders in the matrices a, b, g, respectively, and the superscript T indicates transposition.

These abbreviated equations are derived from (1.1) by dropping certain terms, on the grounds that they are "small". The dynamics of the initial system (1.1) is then investigated using the abbreviated Eqs.(1.2), (1.3) or (1.4). However, whereas the legitimacy of Eqs.(1.2) has been discussed in the literature and there are numerous results for special cases /6, 11-14/, no such treatment exists for equations of type (1.3) and (1.4).

---

*Prikl.Matem.Mekhan.,52,6,915-921,1988

We pose the following problem: it is required to construct simplified models by rigorous means and rigorously establish their legitimacy.

2. We consider a gyroscopic system, assuming that its gyroscopes are high-speed (the internal angular momenta of the gyroscopes may be as large as desired). In terms of system (1.1) this means that the gyroscopic coefficients depend on a large non-dimensional positive parameter H (all other terms in (1.1) are of zero order of magnitude relative to H): $g = g^*H$, $H = 1/\mu$, and μ a small parameter. We now transform (1.1) to a system of differential equations whose coefficients are small parameters. Put $\tau = \mu t$ and introduce new variables $\mathbf{x}_1 = ad\mathbf{q}_M/d\tau$, $\mathbf{x}_2 = \mathbf{q}_M$. Then system (1.1) becomes

$$\mu^2 d\mathbf{x}_1/d\tau = \mathbf{X}_1(\mu, \mathbf{x}), \quad d\mathbf{x}_2/d\tau = \mathbf{X}_2(\mu, \mathbf{x}) \tag{2.1}$$

Transformation to singularly perturbed equations yields a natural way to choose a simplified system and rigorously establish its legitimacy, by separating the motions (and variables) of the system into components of different orders of magnitude.

As a first approximate (abbreviated) system for (2.1), we take the system linearized with respect to μ. Call it system A (this is an abbreviated system of the first kind). In the old variables, system A corresponds to model (1.2), for which we retain the adjective "precessional".

Another possible approximation to system (2.1) is obtained by taking the degenerate system (corresponding to $\mu = 0$), as it is customarily called in the theory of singular perturbations. We call it system B (this is an abbreviated system of the second kind, in Tikhonov's terminology /15/). In the old variables it corresponds to model (1.3). We shall call it the limiting model (as it corresponds to the degenerate system, $\mu = 0$). Models of type (1.3) were considered in /8/, and the possibility of utilizing them as simplified sets of equations was pointed out in /6/.

System (1.4) cannot be derived from (1.1) under the assumptions we have made concerning the physical nature of the model (high-speed gyroscopes). As rightly observed in /6, 7/, its rigorous derivation (and proof) require different physical premises and another approach to the introduction of the small parameter.

3. We will now consider a gyroscopic system with high-speed gyroscopes, whose state is determined by the same n generalized coordinates and by u additional coordinates; it is assumed that the differential equations for the latter do not contain terms involving the large parameter H. This type of system was studied in /11, 16/. By the same method as before, we construct simplified models in a rigorous manner. The study will be illustrated by the example of a gyroscopic stabilizing system (GSS), simulated by an electromechanical system with n mechanical generalized (Lagrangian) coordinates and u electrical generalized (Maxwellian) coordinates /17/. The differential equations of the perturbed motion are as follows (written in the Lagrange-Maxwell or Gaponov form):

$$\begin{aligned} &da\mathbf{q}_M^{\cdot}/dt + (b^\circ + g^\circ)\mathbf{q}_M^{\cdot} = \mathbf{Q}_M'(\mathbf{q}_M) + \mathbf{Q}_{ME}(\mathbf{q}_E^{\cdot}) + \mathbf{Q}_M''(\mathbf{q}_M, \mathbf{q}_M^{\cdot}, \mathbf{q}_E^{\cdot}) \\ &dL\mathbf{q}_E^{\cdot}/dt + R^\circ\mathbf{q}_E^{\cdot} = \mathbf{Q}_E'(\mathbf{q}_M, \mathbf{q}_E^{\cdot}) + \mathbf{Q}_{EM}(\mathbf{q}_M^{\cdot}) + \mathbf{Q}_E''(\mathbf{q}_M, \mathbf{q}_M^{\cdot}, \mathbf{q}_E^{\cdot}), \\ &d\mathbf{q}_M/dt = \mathbf{q}_M^{\cdot} \end{aligned} \tag{3.1}$$

Here

$$\mathbf{Q}_{ME}(\mathbf{q}_E^{\cdot}) = A_M\mathbf{q}_E^{\cdot}, \quad A_M = \|0, A, 0\|^T, \quad \mathbf{Q}_{EM}(\mathbf{q}_M^{\cdot}) = B_E\mathbf{q}_M^{\cdot}$$
$$B_E = \|0, B, 0\|, \quad \mathbf{Q}_E'(\mathbf{q}_M, \mathbf{q}_E^{\cdot}) = \Omega_E\|\mathbf{q}_M, \mathbf{q}_E^{\cdot}\|^T$$

where $\mathbf{q}_E$ is the u-dimensional vector of Maxwellian coordiantes, L is the symmetric $(u \times u)$ matrix of the positive-definite form representing the electromagnetic energy of the system, $R(\mathbf{q}_E^{\cdot})$ is the symmetric $(u \times u)$ matrix of the positive-definite quadratic form in the decomposition of the dissipative flux function characterizing the Joule heat loss, $\mathbf{Q}_{ME}$ and $\mathbf{Q}_{EM}$ are mechanical generalized forces of electrical origin (ponderomotive forces) and electrical generalized forces of mechanical origin, $A = \|A_{kj}\|$ is an $(s-m) \times u$ matrix, $B = \|B_{kj}\|$ is a $u \times (s-m)$ matrix, $\mathbf{Q}_E'$ is the vector of electrical generalized forces corresponding to the electrical generalized coordinates, Ω_E is a $u \times (n+u)$ matrix, and $\mathbf{Q}_M''$ and $\mathbf{Q}_E''$ are sums of non-linear terms. System (3.1) is of order $(2n+u)$.

In a system with high-speed gyroscopes, as in Sect.2, we have $g = g^*H, H = 1/\mu$. Put $\tau = \mu t$ and define new variables

$$\mathbf{x}_1 = a\,d\mathbf{q}_M/d\tau, \quad \mathbf{x}_2 = L\mathbf{q}_E^{\cdot}, \quad \mathbf{x}_3 = \mathbf{q}_M$$

In terms of these new variables, system (3.1) becomes

$$\begin{aligned} &\mu^{\alpha_k} d\mathbf{x}_k/d\tau = \mathbf{X}_k(\mu, \mathbf{x}) \ (k = 1, 2, 3) \\ &\alpha_1 = 2, \ \alpha_2 = 1, \ \alpha_3 = 0 \end{aligned} \tag{3.2}$$

Using separation of variables into components of different classes, one can construct various natural simplified systems. The simplified system of the first kind, obtained by linearization (with respect to the small parameter μ), corresponds in the old variables to a system of equations of order $(n+u)$:

$$(b^\circ + g^\circ)\dot{\mathbf{q}}_M = \mathbf{Q}_M'(\mathbf{q}_M) + \mathbf{Q}_{ME}(\mathbf{q}_E^{\cdot}) + \overline{\mathbf{Q}}_M''(\mathbf{q}_M, \dot{\mathbf{q}}_M, \mathbf{q}_E^{\cdot})$$
$$dL\mathbf{q}_E^{\cdot}/dt + R^\circ\mathbf{q}_E^{\cdot} = \mathbf{Q}_E'(\mathbf{q}_M, \mathbf{q}_E^{\cdot}) + \mathbf{Q}_{EM}(\dot{\mathbf{q}}_M) + \mathbf{Q}_E''(\mathbf{q}_M, \dot{\mathbf{q}}_M, \mathbf{q}_E^{\cdot})$$
$$d\mathbf{q}_M/dt = \dot{\mathbf{q}}_M \tag{3.3}$$

We shall call this the precessional model (as in /11, 16/).

The simplified system of the second kind (corresponding to an approximate system accurate to within μ^0, i.e., the degenerate system), gives a system of order n:

$$g^\circ\dot{\mathbf{q}}_M = \mathbf{Q}_M'(\mathbf{q}_M) + \mathbf{Q}_{ME}(\mathbf{q}_E^{\cdot}) + \overline{\mathbf{Q}}_M''(\mathbf{q}_M, \dot{\mathbf{q}}_M, \mathbf{q}_E^{\cdot})$$
$$R^\circ\mathbf{q}_E^{\cdot} = \mathbf{Q}_E'(\mathbf{q}_M, \mathbf{q}_E^{\cdot}) + \overline{\mathbf{Q}}_E''(\mathbf{q}_M, \dot{\mathbf{q}}_M, \mathbf{q}_E^{\cdot}), \quad d\mathbf{q}_M/dt = \dot{\mathbf{q}}_M \tag{3.4}$$

We shall call this the limiting model.

Thus, with regard to the systems under consideration here (of types (2.1) and (3.2)), the theory of singular perturbations enables us to single out a sequence of approximate (simplified) systems, which (as it applies to gyroscopic systems) permit of a clear physical interpretation. *To be specific, the simplified system of the first kind corresponds to the* traditional precessional model ((1.2) and (3.3)); the simplified system of the second kind corresponds to the simpler limiting model ((1.3) and (3.4)).

4. We must now specify conditions under which the transition to a simplified model of lower order is legitimate. Our problem is as follows: under what conditions does stability of the simplified model guarantee that the initial system has the same stability property, and under what conditions are the solutions of the simplified and the full system similar to one another over an infinite time interval? Following an idea out forward by N.G. Chetayev /4/ and using the methods of stability theory, one can establish conditions of this kind (as in /17/).

Theorem 1. Let $|g^\circ| \neq 0$ and suppose that all the roots of the characteristic equation of the simplified system (1.2) lie in the left half-plane (with the possible exception of m *zero roots in the case of a GSS), and that the equation* $|a^\circ\lambda + b^\circ + g^\circ| = 0$ *satisfies the* Hurwitz conditions. Then for sufficiently large H, if the trivial solution of the simplified (precessional) model is asymptotically stable (stable), then the same is true of the trivial solution of the complete (initial) system (1.1). Given any prescribed numbers $\varepsilon > 0$, $\delta > 0$, $\gamma > 0$ (ε and γ may be as small as desired), there exists H_* such that for all $H > H_*$ and all $t \geqslant t_0 + \gamma$, the inequalities

$$\|\dot{\mathbf{q}}_M - \dot{\mathbf{q}}_M^s\| < \varepsilon, \quad \|\mathbf{q}_M - \mathbf{q}_M^s\| < \varepsilon$$

hold throughout the perturbed motion, provided that at time t_0

$$\|\dot{\mathbf{q}}_{M0} - \dot{\mathbf{q}}_{M0}^s\| < \delta, \quad \mathbf{q}_{M0} = \mathbf{q}_{M0}^s$$

The superscript "s" indicates that the solution is of the simplified system: $\dot{\mathbf{q}}_M^s = \boldsymbol{\varphi}(\mathbf{q}_M^s)$, where $\dot{\mathbf{q}}_M = \boldsymbol{\varphi}(\mathbf{q}_M)$ is the solution of the algebraic equation in (1.2) for $\dot{\mathbf{q}}_M$.

Proof. In accordance with the approach adopted here, we consider (1.1) as a singularly perturbed system. In the case of high-speed gyroscopes, Eqs.(1.1) reduce to the form of (2.1), where $\mathbf{X}_j(\mu, \mathbf{x}) = P_j(\mu)\mathbf{x} + \ldots$ $(j = 1, 2)$. When this is done, system (2.1) is not approximated by the traditional system but by that linearized with respect to μ (system A). Note that systems (2.1) and A are special cases of the more general systems (1.1) and (1.2) considered in /17/, for which theorems have been established concerning the stability and proximity of the solutions in an infinite time interval, as well as the appropriate estimates /17/. Using Theorems 1 and 2 of /17/, we infer the following properties of systems (2.1) and A: 1) if $|P(0)| \neq 0$ and the equation $|\beta E - P_{11}(0)| = 0$ satisfies the Hurwitz conditions, and moreover all the roots of the characteristic equation of the simplified system A lie in the left half-plane (with the possible exception of m zero roots), then for sufficiently small μ asymptotic stability (stability) of the trivial solution of the approximate system A implies the parallel property of the complete system (2.1); 2) for preassigned values of the positive numbers ε, δ, γ (where ε and γ may be as small as desired), there exists μ_* such that for $0 < \mu < \mu_*$ and all $t > t_0 + \gamma$ the perturbed motion satisfies the inequality $\|\mathbf{x} - \mathbf{x}^*\| < \varepsilon$, provided that initially $\|\mathbf{x}_{10} - \mathbf{x}_{10}^*\| < \delta$, $\mathbf{x}_{20} = \mathbf{x}_{20}^*$. The asterisk indicates a solution of the approximate system A; letters without an asterisk denote solutions of the complete system (2.1). Returning to the old variables $\mathbf{q}_M, \dot{\mathbf{q}}_M$ and noting that the transformation we have used is non-linear, non-singular, uniformly regular and preserves stability, we obtain the assertions of Theorem 1.

An analogous result has been obtained for the limiting model (1.3). The appropriate assertions are also true for GSS (system (3.1)).

Our investigations furnish conditions under which the simplified model is admissible (in the sense used here).

5. We will consider the legitimacy of models of type (1.4). The previous physical assumptions (high-speed gyroscopes) are no longer appropriate. To choose (1.4) as a simplified model, we shall assume that the equatorial moments of inertia of the gyroscopes and the moments of inertia of the suspensions (and their masses) are small compared with the mass characteristic of the objects (platforms) on which the gyroscopes are standing. In this connection we put $a_1 = a_1(\mathbf{q}_M, \mu) = a_1{}^*\mu$ in (1.1), where $\mu > 0$ is a small non-dimensional parameter, $a_2 = a_2(\mathbf{q}_M, \mu)$, where $a_2(\mathbf{q}_M, 0) = \bar{a}_2 \neq 0$.

In accordance with the technique proposed here, we replace Eqs.(1.1) by singularly perturbed equations. To this end we introduce the new variables $\mathbf{x}_1 = a_1{}^*\mathbf{q}_M{}^{\cdot}$, $\mathbf{x}_2 = \| a_2\mathbf{q}_{\dot{M}}, \mathbf{q}_{\dot{M}} \|^T$, in terms of which system (1.1) becomes

$$\mu\, d\mathbf{x}_1/dt = \mathbf{X}_1(\mu, \mathbf{x}), \quad d\mathbf{x}_2/dt = \mathbf{X}_2(\mu, \mathbf{x}) \tag{5.1}$$

Putting $\mu = 0$ in this equation and taking the degenerate system as an approximation of (5.1), then returning to the old variables, we obtain system (1.4). It is perfectly natural to call this simplified system (1.4) the limiting model, since it corresponds to the degenerate system for (5.1) (but in a different sense than in Sect.2 and 3, since we have introduced a different small parameter).

From the standpoint of mechanics, here too, as before, we have used the idea of separating different motions, thus obtaining a system (1.4) corresponding to a mechanical system with fewer degrees of freedom.

6. We now need conditions under which it is admissible to use the simplified model (1.4) to investigate the dynamics of system (1.1). By the same method, following N.G. Chetayev and using the results of stability theory, we can prove the following assertions.

Theorem 2. If $|e^\circ| \neq 0$ and the equations

$$\begin{vmatrix} (b_1{}^\circ + g_1{}^\circ)\lambda + e_1{}^\circ \\ \bar{a}_2{}^\circ\lambda^2 + (b_2{}^\circ + g_2{}^\circ)\lambda + e_2{}^\circ \end{vmatrix} = 0, \quad \begin{vmatrix} a_1{}^*\alpha + b_1{}^\circ + g_1{}^\circ \\ \bar{a}_2{}^\circ \end{vmatrix} = 0 \tag{6.1}$$

satisfy the Hurwitz conditions (or, in the case of GSS, the first of Eqs.(1.1) may have m zero roots), then for sufficiently small values of the parameter μ asymptotic stability (stability) of the trivial solution of the limiting system (1.4) implies the parallel property of the trivial solution of system (1.1). For prescribed values of $\varepsilon > 0, \delta > 0, \gamma > 0$ (ε and γ may be as small as desired), there exists a value of μ_* such that in the perturbed motion, for all $0 < \mu < \mu_*$ and all $t \geqslant t_0 + \gamma$,

$$\| \dot{\mathbf{q}}_M - \dot{\mathbf{q}}_M^s \| < \varepsilon, \quad \| \mathbf{q}_M - \mathbf{q}_M^s \| < \varepsilon$$

provided that at t_0

$$\| \dot{\mathbf{q}}_{10} - \dot{\mathbf{q}}_{10}^s \| < \delta, \quad \dot{\mathbf{q}}_{20} = \dot{\mathbf{q}}_{20}^s, \quad \mathbf{q}_{M0} = \mathbf{q}_{M0}^s$$

(the index s indicates a solution of the simplified system (1.4), with $\mathbf{q}_1{}^{\cdot s} = \boldsymbol{\varphi}_1(\mathbf{q}_2{}^{\cdot s}, \mathbf{q}_M{}^s)$, where $\mathbf{q}_1{}^{\cdot} = \boldsymbol{\varphi}_1(\mathbf{q}_2{}^{\cdot}, \mathbf{q}_M)$ is the solution of the algebraic equation of (1.4) in $\mathbf{q}_1{}^{\cdot}$.

The proof uses the same approach as that of Theorem 1. Note that system (5.1) in the new variables is also a special case of the systems used in /17/.

This result gives a rigorous justification of a simplified model widely used for computations in applied research /9, 10/. Under the conditions derived above, it is admissible (in the above sense) to go over to that model and utilize it as a working model to investigate the physical system in question.

7. As examples, we investigated gyroscopic stabilizing systems. Here we shall consider a uniaxial gyrostabilizer (UGS), simulating it as an electromechanical system /18/, taking into account transients in the electric circuits of the servomechanisms. As in /7/, we assume that the stabilizer is an independently driven DC motor with controlled armature current.

The different equations of the perturbed motion /18/ form a seventh-order system ($n = 2$, $u = 3$) of type (3.1):

$$B\beta^{\cdot\cdot} - H\alpha^{\cdot} + b_1\beta^{\cdot} = \ldots, \quad \bar{J}\alpha^{\cdot\cdot} + H\beta^{\cdot} + b_2\alpha^{\cdot} = g_M i_2 + \ldots$$

$$\sum_{j=1}^{3} L_{kj} i_j{}^{\cdot} + R_k i_k = E_k + \ldots \quad (k = 1, 2, 3)$$

$$E_1 = -\omega\beta, \quad E_2 = -\Omega i_1 - g_E\alpha^{\cdot}, \quad E_3 = 0$$

We have retained the notation of /18/; β is the angle of rotation of the gyroscope casing relative to the frame (the angle of precession), α is the angle of rotation of the frame (the angle of stabilization), i_1, i_2, i_3 are the currents in the amplifier, armature and field winding currents, respectively, L_{kj} $(k, j = 1, 2, 3)$ are the coefficients of self and mutual inductance in these circuits, R_k $(k = 1, 2, 3)$ are the resistances, b_1 and b_2 are the coefficients of viscous friction on the axes of the suspensions of the gyroscope casing and frame, respectively, B and J are the moment of inertia of the gyroscope and the reduced moment of inertia of the gyrostabilizer relative to the appropriate axes, g_M and g_E the coefficients in the expressions for the mechanical forces of electrical origin and the electrical forces of mechanical origin, respectively, and ω and Ω the coefficients in the expressions for the electrical generalized forces; the ellipses points stand for the omitted non-linear terms.

The results of Sects.2, 3 imply that in the case of high-speed gyroscopes we can construct two kinds of working model: the precessional model (type (3.3)), in this case a fifth-order system of equations:

$$-H\alpha^{\cdot} + b_1\beta^{\cdot} = \ldots, \quad H\beta^{\cdot} + b_2\alpha^{\cdot} = g_M i_2 + \ldots$$
$$\sum_{j=1}^{3} L_{kj} i_j^{\cdot} + R_k i_k = E_k + \ldots \quad (k = 1, 2, 3)$$

and the limiting model (type (3.4)), in this case a second-order system:

$$-H\alpha^{\cdot} = \ldots, \quad H\beta^{\cdot} = g_M i_2 + \ldots$$
$$R_1 i_1 = -\omega\beta + \ldots, \quad R_2 i_2 = -\Omega i_1 + \ldots, \quad R_3 i_3 = \ldots$$

It follows from our results that if H is sufficiently large (a sufficiently large value of the intrinsic angular momentum of the gyroscope), the precessional model of the OGS is admissible (in the sense adopted here), provided the trivial solution of the simplified system is stable and the quadratic form of the dissipative function, corresponding to the mechanical generalized coordinates, is positive-definite.

However, if the mass of the stabilized object is large compared with that of the gyroscope, a different model must be used /6, 7/. By Sect.5, we see that this is a limiting model of type (1.4), corresponding here to the sixth-order system

$$-H\alpha^{\cdot} + b_1\beta^{\cdot} = \ldots, \quad J\alpha^{\cdot\cdot} + H\beta^{\cdot} + b_2\alpha^{\cdot} = g_M i_2 + \ldots$$
$$\sum_{j=1}^{3} L_{kj} i_j^{\cdot} + R_k i_k = E_k + \ldots \quad (k = 1, 2, 3)$$

Transition to this system is admissible at sufficiently small values of the moment of inertia of the gyroscope relative to the suspension axis of the housing, provided the trivial solution of the simplified system is stable and the torque exerted by viscous friction on the axis of precession is not zero $(b_1 > 0)$.

REFERENCES

1. BOGOLYUBOV N.N. and MITROPOL'SKII YU.A., Asymptotic Methods in the Theory of Non-linear Oscillations, Nauka, Moscow, 1974.
2. KUZ'MINA L.K., Methods of stability theory and singularly perturbed systems in mechanics. In: Proceedings of the 6th All-Union Conference on Theoretical and Applied Mechanics, Tashkent, National Committee of the USSR on Theoretical and Applied Mechanics, 1986.
3. LYAPUNOV A.M., The general problem of stability of motion. In: Collected Works, 2, Izd. Akad. Nauk SSSR, Moscow, 1956.
4. CHETAYEV N.G., On the question of estimates of approximate integrations. Prikl. Mat. Mekh., 21, 3, 1957.
5. VASIL'YEVA A.B. and BUTUZOV V.F., Asymptotic Expansions of the Solutions of Singularly Perturbed Equations, Nauka, Moscow, 1973.
6. MERKIN D.R., Gyroscopic Systems, Gostekhizdat, Moscow, 1956.
7. ISHLINSKII A.YU., Orientation, Gyroscopes and Inertial Navigation, Nauka, Moscow, 1976.
8. METELITSYN I.I., On the problems of gyroscopic stabilization. Dokl. Akad. Nauk SSSR, 86, 1, 1952.
9. RAUSHENBAKH B.V. and TOKAR E.N., Control of the Orientation of Space Ships, Nauka, Moscow, 1974.
10. SARYCHEV V.A., MIRER S.A. and ZLATOUSTOV V.A., Optimal parameters of an aero-gyroscopic system for orientation of satellites. Kosm. Issled, 22, 3, 1984.
11. NOVOSELOV V.S., The motion of gyroscopic systems. Prikl. Mat. Mekh., 23, 1, 1959.
12. ZUBOV V.I., Analytical Dynamics of Gyroscopic Systems, Sudostroyeniye, Leningrad, 1970.
13. KOBRIN A.I., MARTYNENKO YU.G. and NOVOZHILOV I.V., On the precessional equations of gyroscopic systems. Prikl. Mat. Mekh., 40, 2, 1976.

14. SOBOLEV V.A. and STRYGIN V.V., On the admissibility of the transition to the precessional equations of gyroscopic systems. Izv. Akad. Nauk SSSR, MTT, 5, 1978.
15. TIKHONOV A.N., Systems of differential equations containing small parameters as coefficients of the derivatives. Mat. Sbornik, 31, 3, 1952.
16. NOVOSELOV V.S., The motion of non-linear gyroscopic systems. Prikl. Mat. Mekh., 24, 6, 1960.
17. KUZ'MINA L.K., On some properties of the solutions of singularly perturbed systems in a critical case. Prikl. Mat. Mekh., 46, 3, 1982.
18. KUZ'MINA L.K., On the stability of gyrostabilizers. In: Proceedings of the 1st Volga Conference of Automatic Control, Kazan', Tatar. Kn. Izd., Book 1, 1971.

Translated by D.L.

PMM U.S.S.R.,Vol.52,No.6,pp.720-724,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

CONTINUOUS MODAL CONTROL OF LINEAR MULTICOUPLED OBJECTS*

V.A. BRUSIN and YU.M. MAKSIMOV

A modal control method is considered in which the spectrum of the closed-loop system is continuously deformed in such a way that the spectrum of the open-loop object transforms into the desired spectrum. The algorithm of the continuous modal control is synthesized. The conditions for spectral control in the method are obtained. The approach is based on similar ideas to those in /1/, but a different class of controls is considered here. Moreover, by using the appratus of Lyapunov functions, specified in the one-parameter family of the deformed spectrum, the deviation between the required spectrum and the closed-loop system spectrum can be minimized in the Euclidean metric, in the case when the wanted spectrum cannot be obtained in the closed-loop system.

1. Formulation of the problem.

Suppose we are given the linear controlled object

$$x^{\cdot}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \tag{1.1}$$
$$x \in R^n, \quad u \in R^m, \quad y \in R^l$$

where x is the state vector, u is the control vector, y is the vector of observed variables A, B, C are constant matrices of suitable dimensionless, and R^n is a linear n-dimensional space over the real number field. We shall in future assume that the spectrum of the object (1.1) is simple and contains no multiple poles. We define the class of controls by

$$u_\alpha(t) = \left(\int_0^\alpha G(\xi)\,d\xi\right) y(t), \quad G \in R^{m\times l} \tag{1.2}$$

where G is a matrix function of the scalar variable ξ, and $\alpha \geqslant 0$ is a parameter. The dynamic behaviour of the closed-loop system is given by the matrix

$$A(\alpha) = A + B\left(\int_0^\alpha G(\xi)\,d\xi\right) C \tag{1.3}$$

whose spectrum is a function of the parameter α. With $\alpha = 0$ we have the open-loop system, whose spectrum is denoted by $\Lambda(0)$. As α varies, the class of linear systems is generated. Every element of the class (the linear system which has the spectrum $\Lambda(\alpha) = \{p_1(\alpha), p_2(\alpha), \ldots, p_n(\alpha)\}$) is defined by a specific value of the parameter α.

**Prikl.Matem.Mekhan.*,52,6,922-928,1988